

Application of Fractals in Telecommunications: Yahweh-Emanuel Vaz's Fractal Patch Antenna

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ABSTRACT: The Vaz fractal is presented with an appropriate calculation treatment. The Vaz fractal is applied to the reduction of antenna length. Vaz fractal antennas exhibit lower resonance frequencies than linear dipoles of equal length and are therefore suitable for miniaturization of equipment. Studies on the subject are widely disseminated in journals. From linear log-periodic modeling, a fractal log-periodic antenna with Emanuel Vaz fractal dipoles was obtained and studied with three iterations. This article analyzes the Emanuel Vaz fractal antenna in its first three iterations, with a total height of 20 cm. Through simulations with the 4NEC2 program, the parameters of resonance frequency, input impedance, standing wave ratio and radiation pattern obtained are compared with those of a linear dipole of the same height. The antennas resulting from the first two iterations and the linear dipole antenna were fabricated with copper wire, while the antenna resulting from the third iteration was etched in copper onto a rigid cardboard plate due to its geometric complexity. The results obtained in the measurements show good agreement with simulations and other studies of fractals as antennas.

KEYWORDS: Antenna, Fractal, Iterations, log-periodic antenna.

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1. INTRODUCTION

Emanuel Vaz's fractals can be obtained from a line by dividing it into three equal parts and, in the middle, constructing the three remaining sides of a square as seen in Figure 1. The same thing happens for any segment, and successive fractal figures E-2, E-3, etc. are generated.

The compactness of mobile communication devices requires antennas smaller than those previously used. This is one of the applications of fractal antennas, which are either small or multiband, and can be used for cellular telephony, RFID, Wi-Fi, digital TV, among other applications, ref 10. Antennas with fractal elements, can be multiband, due to the characteristic of self-similarity, which can be described as the replication of the geometry at different scales, or they can be compact, due to the space-filling property, as is the case with antennas that use certain important curves. Fractal curves have been applied as monopole, loop, microstrip, and logperiodic antennas. Throughout this article, the results of simulations performed with Emanuel Vaz dipole antennas with a total height of 20 cm are presented, and parameters such as resonance frequencies, input impedance, standing wave ratio, and radiation pattern of the antennas are compared with those of a linear dipole of the same height.

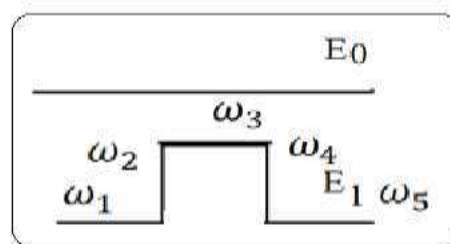


Fig.1. Development of the Emanuel Vaz fractal

2. FRACTAL GEOMETRY

The term fractal was coined by the mathematician Mandelbrot in the 1970s after his pioneering research into various naturally irregular and fragmented geometries not contained within conventional Euclidean geometry, ref 8,9. Mandelbrot formulated hypotheses according to which many of these geometries are infinitely subdivisible, where each division is a copy of the whole. Fractals can be used to classify structures whose dimensions are not integers. Fractal geometries can be generated iteratively, mainly from self-similar structures. Fractal geometry has great potential to increase antenna performance. The property of self-similarity makes certain fractal geometries suitable for designing multiband antennas that operate similarly for different frequencies, exhibiting various resonance frequencies, as is the case with the antenna based on Sierpinski geometry. Fractals also exhibit the property of allowing greater space fill, enabling the design of antennas with greater electrical length in a much more delimited physical space. This makes it possible to build small antennas that occupy the same volume as Euclidean antennas, but are much longer electrically. The increased electrical length leads to a lower resonant frequency, which can be used to miniaturize the antenna elements.

3. EMANUEL VAZ'S FRACTAL CURVE

Emanuel Vaz geometry was originally studied for successful application in the fabrication of fractal antennas. Figure 1 shows the first four iterations of Emanuel Vaz's fractal curve, which have the same height h , that is, the same linear distance between the first and last point. The first element of the series is a linear segment called E_0 (zero iteration of the fractal construction) which represents the Euclidean linear monopole. The sequential iterations are named E_1, E_2, E_3, E_4 .

To obtain the first iteration of Emanuel Vaz's fractal curve, a linear segment is divided into three parts of equal length, then the middle segment is replaced by three segments equal to this, forming a square without a base. Each segment of the first iteration, E_1 , has one-third the length of E_0 , so now there are four straight lines with lengths of $\frac{1}{3}$ of the original linear segment, totaling a length that is $\frac{5}{3}$ of the length of E_0 . At each iteration, the shape E_1 is inserted into the straight segments, on a scale three times smaller than the previous one. The total length of Emanuel Vaz's fractal curve, l_f , grows by $\frac{5}{3}$ at each iteration, so the final length of the wire at each n iterations is given by the relation:

$$l_f = \left(\frac{5}{3}\right)^n \times l_0 \quad (1)$$

A classic algorithm for constructing Emanuel Vaz's fractal curve is the Iterated Function System (IFS) algorithm, which can be applied to generate successive curves that converge to an ideal fractal shape. An IFS is based on applying a series of affine transformations ω to an elementary form, ref^a 9. The logic of this function is presented below.

An affine transformation can be written as follows.

$$\omega \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} r_1 \cos \theta_1 & -r_2 \sin \theta_2 \\ r_1 \sin \theta_1 & r_2 \cos \theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad (2)$$

Therefore, the affine transformation is represented by six parameters where a, b, c , and d control the scale and rotation, while e and f control the linear translation. In the special case where, there is a similarity in which represents the scale factor and the rotation angle. Given the straight segment, E_0 , situated along the x -axis with its left endpoint at the origin, the segments to obtain the first iteration E_1 are reduced by a factor of $\frac{1}{3}$, have rotations of $0^\circ, 90^\circ, 0^\circ, -90^\circ$, and 0° respectively, and are indicated in Fig. 1. Substituting these parameters into the transformation equation yields the following equations, ref 12.

$$\omega_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cos 0^\circ & -\frac{1}{3} \sin 0^\circ \\ \frac{1}{3} \sin 0^\circ & \frac{1}{3} \cos 0^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

$$\omega_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cos 90^\circ & -\frac{1}{3} \sin 90^\circ \\ \frac{1}{3} \sin 90^\circ & \frac{1}{3} \cos 90^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \quad (4)$$

$$\omega_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cos 0^\circ & -\frac{1}{3} \sin 0^\circ \\ \frac{1}{3} \sin 0^\circ & \frac{1}{3} \cos 0^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad (5)$$

$$\omega_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cos(-90^\circ) & -\frac{1}{3} \sin(-90^\circ) \\ \frac{1}{3} \sin(-90^\circ) & \frac{1}{3} \cos(-90^\circ) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad (6)$$

$$\omega_5 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cos 0^\circ & -\frac{1}{3} \sin 0^\circ \\ \frac{1}{3} \sin 0^\circ & \frac{1}{3} \cos 0^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} \quad (7)$$

The case that interests us here is exactly one-dimensional, but the fractals in the antennas can easily be taken in three dimensions using an array factor that allows the calculation of important parameters in three dimensions.

4. FRACTAL DIMENSION OF THE EMANUEL VAZ'S CURVE

Conventional geometric figures have a determined dimension. Lines have dimension 1, the plane has dimension 2, and the cube has dimension 3. This is the topological dimension, which is always expressed as an integer and characterizes Euclidean geometry. Mandelbrot introduced the idea that irregular curves have a dimension between one and two, and irregular surfaces have a dimension between two and three. The fractal dimension can be represented by a fraction. The most easily understood fractal dimension is that of self-similarity. To obtain this value, the geometry is divided into identical copies of itself on a reduced scale. If there are N copies of the original geometry reduced by a fraction s , the fractal similarity dimension D is defined by the following expression:

$$D = \frac{\log N}{\log \left(\frac{1}{s}\right)} \quad (8)$$

Since in each iteration there are 5 copies of the original geometry with a size 3 times reduced, then $\frac{5}{3}$. Thus, the dimension D of Emanuel Vaz's curve is calculated by the following result.

This is easily seen in light of the theory by examining the matrices written previously; it is enough to verify that the second matrix of each affine transformation, here, is $\frac{1}{3}$ of the identity matrix.

$$D = \frac{\log N}{\log \left(\frac{1}{s}\right)} = \frac{\log 5}{\log 3} = 1,4650 \quad (9)$$

5. EMANUEL VAZ FRACTAL DIPOLE

The simulation in 4NEC2. The linear dipole antenna and the dipoles with the first three iterations of Emanuel Vaz's curve, from that point on related as E_0, E_1, E_2, E_3 , respectively represent dipoles with the same total height h , of 20 cm, while the total wire lengths are respectively: 20 cm, 33.33 cm, 55.55 cm, and 92.59 cm, according to the equation we derived. The dipoles were simulated in 4NEC2, software based on the method of moments that analyzes the electromagnetic response of arbitrary structures constructed with thin wires or surfaces. The simulated antennas are made of copper wire and immersed in free space. All simulated dipole antennas are center-fed with the same total height h equal to 20 cm. Dipoles E_0, E_1, E_2 , were modeled in 4NEC2 with copper wire radii of 0.0001 m, and dipole E_3 was modeled with a radius of 0.00001 m. For fast computational tracking, a formula for the straight divisions of the E_3 model is deduced as a decrement, which we will present at the end of the article.

Table I relates the resonant frequency, input resistance, and SWR values (relative to an impedance of 75 Ω) obtained from the simulations of the linear dipole and the first three iterations of Emanuel Vaz's fractal dipole with heights of 20 cm. It can be observed that with each iteration the resonant frequency decreases, as does the input resistance at the resonant frequency.

6. SIMULATIONS

The arrays were simulated in "4NEC2, free software modernizer and optimizer", ref 11, based on the method of moments studied in the text of the book Antennas and Propagation, ref 1,3, and analyzes the electromagnetic response of arbitrary structures built with thin wires or surfaces. The simulated arrays consist of copper wire immersed in free space. The antenna array is fed at the center of the dipoles, which are connected by a transmission line. For each array, six dipoles were designed; the LDPA was designed with 12 points and the LPFKA-E3 with 768 points in total. The straight divisions were discretized according to the autosegmentation of the software itself, and the radii were stipulated so that there was no numerical error.

Firstly, this article designed an LDPA array in which the lengths and spacings of the linear dipoles were calculated from equations obtained from the Patch antenna chapter of the book Propagation Antennas, referenced at the end of the text, to achieve a minimum resonance frequency of 470 MHz.

One of the important design parameters of a log-periodic antenna is the scale factor (τ). The relationship between the different dipoles that make up the array and the scale factor is represented by the following equation where l_n represents the length of each dipole n in the array, and d_n represents the distance between the dipoles. In this work, τ was adopted as 0.9, as this is the value that provided the lengths of the dipoles that result in the antennas with the best return loss value in the desired frequency range among the various models analyzed.

$$\tau = \frac{l_n}{l_{n+1}} = \frac{d_n}{d_{n+1}} \quad (10)$$

Based on observations from the simulations, in the best-performing LPFKA-E3 array, the l_n lengths of each E_3 dipole for the linear array are in the ratio of 68%/1, corresponding to a percentage reduction of 32%, while the d_n spacings have a ratio very close to that, to achieve resonance starting at 470 MHz.

Table 1. Dimensions of the designed arrays				
	LPDA	LPEKA-E3	LPDA	LPEKA-E3
Dipoles N	Length of the dipoles l_n (m)	Distance between the dipoles d_n (m)	Length of the dipoles l_n (m)	Distance between the dipoles d_n (m)
<u>1</u>	0,31915	1,4775	0,21702	1,0056
<u>2</u>	0,28723	0,7979	0,19532	0,5456
<u>3</u>	0,25851	1,1968	0,17579	0,8131
<u>4</u>	0,23266	1,7952	0,15821	1,2208
<u>5</u>	0,20939	2,6928	0,14239	1,8311
<u>6</u>	0,18845	4,0391	0,12815	2,7467

The DOTIG6 program was also used in this confirmation to take advantage of studies of the interference of transient electromagnetic waves with arbitrary very thin wire structures. In the work carried out, LDPA log-periodic antennas with linear dipoles and LPFKA-E3 antennas were studied, and their performance in the digital TV frequency range from 470 MHz to 834 MHz was analyzed. The DOTIG6 program uses the method of moments of the electric field integral equation for thin wires (EFIE) and two-dimensional space-time Lagrangian interpolation functions and Dirac delta-type weight functions.

From the linear log-periodic modeling, a fractal log-periodic antenna with Emanuel Vaz fractal dipoles was obtained with three iterations. The miniaturization of the antenna became clear, with a 32% reduction in the length of the dipoles and in the total distance of the LPFKA-E3 array for performance similar to that of the LDPA, at frequencies starting from 470 MHz. On the other hand, there was a slight decrease in gain, analyzed at a frequency of 610 MHz. To optimize the results and make the antennas resonant across the entire frequency range of digital TV, more dipoles can be added to the LDPA and LPFKA-E3 arrays, which will increase the bandwidth of the antennas. Another alternative that can be analyzed is to change the spacing between the dipoles and test the performance of the antennas.

The dependence of the number of iterations, N_{it} , of Emanuel Vaz's fractal antennas on the wire lengths is given by the following expression, ref 10, where h is the straight segment:

$$L_{fio} = h \times \left(\frac{5}{3}\right)^{N_{it}} \quad (11)$$

Since the resonant frequency can be increased by increasing the monopole order, the Emanuel Vaz fractal antenna can be considered small.

To measure the efficiency of the Emanuel Vaz fractal antenna at low frequencies with a quality parameter Q , this is usually given by:

$$Q = \frac{\omega}{2 R_{in}} \left(\frac{dX_{in}}{d\omega} + \left| \frac{X_{in}}{\omega} \right| \right) \quad (12)$$

The smaller Q , the better the efficiency at that frequency. The minimum theoretical value is given by:

$$Q = \frac{1}{(kh)^3} + \frac{1}{kh} \quad (13)$$

For a better understanding of the results, Fig. 2 shows three graphs, which individually represent the SWR, input resistance R_{in} , and input reactance X_{in} values for dipoles E0, E1, E2, and E3 in the frequency range of 300 MHz to 3000 MHz. Only the SWR graph is presented on a logarithmic scale; the others are on a linear scale. It is possible to verify that the resonance frequencies for each dipole are those where X_{in} crosses zero, consequently presenting the lowest SWR and R_{in} values at these frequencies. At the discontinuity points of X_{in} , extremely high SWR and R_{in} values occur. In the first graph, it is noticeable that dipole E3 presents more resonance frequencies in the same frequency range compared to the linear dipole, and this is one of the advantages of Emanuel Vaz's fractal geometry.

Table 2. K-3 Records			
Frequency, MHz	SWR	Rin , Ohms	Xin
470	6	10	0
900	120	20	-2200
1500	105	10000	-6000
1800	60	10	0
2100	80	70	1000
3000	50	3000	-1000

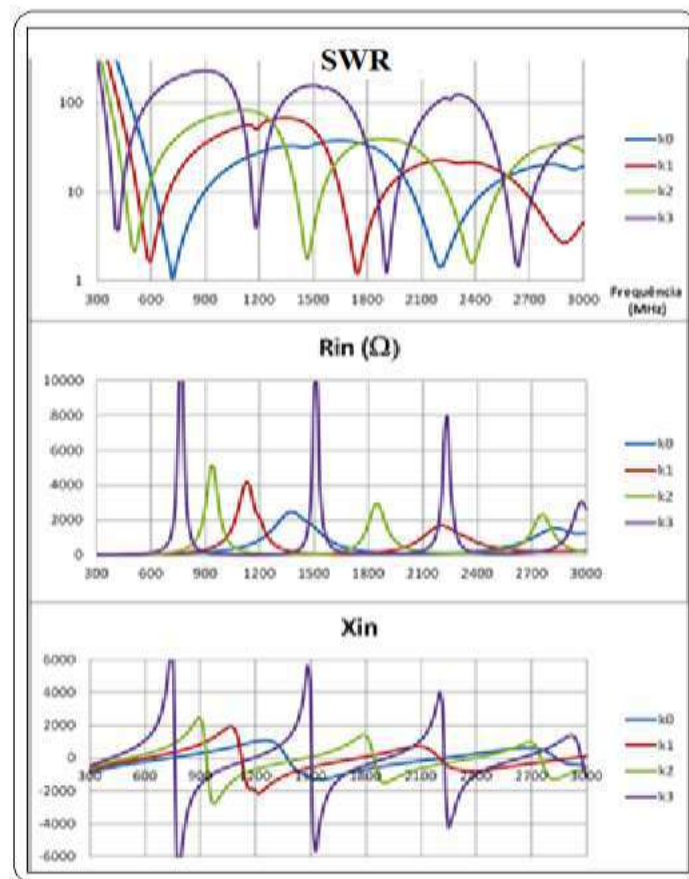


Fig. 2. SWR and Xin graphs for dipoles E1, E2, and E3

7. COMPUTER SPEED

My friend Yahweh at our university challenged me to develop a process capable of determining the ordinates of E3, to adapt it to the program for calculating similar ordinates in Koch antennas. Therefore, it became part of the article's title as a token of gratitude. Once an approximate equation is determined using Sergides' method, we have a path to determining $x = F(f(x))$ knowing ordinates from drawing data or determining $f(x)$ having convenient abscissa points. Thus, it is possible to adjust an equation precisely to determine simulations from E3 upwards.

Sergides' method translates into the use of the following formula and its inherent philosophy.

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)} \text{ for } n = 1, 2, 3, \dots, n, n+1, \dots \quad (14)$$

Using, based on Sergides' method, the sequence of values $g(x_{n-1})$ by $g(x_n)$ and noting that they have the same limit, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_{n-1})} \text{ for } n = 1, 2, 3, \dots, n, n+1, \dots \quad (15)$$

In the case of Newton's iteration problem, $g(x)$ is the derivative $f'(x)$ and the principal value of the logarithm, ref 6, is used.

If the engineer's eye can solve it, to speed up the convergence process, there is the solution; otherwise, the computer or pocket calculator solves it.

The program I present below is part of the BASIC solution to the NewtonRaphson .

$$x_{n+1} - x_n = -\frac{f(x_n)}{g(x_{n-1})}, \Leftrightarrow x_{n+1} - x_n = -\frac{f(x+h)}{g(x)} \quad (16)$$

(for very small values of h)

Using the finite increment theorem, it has been approaching the limits.

$$x_{n+1} - x_n = -\frac{f(x)}{g(x)} - \frac{hf'(x+h)}{g(x)} - \epsilon(h)h \Leftrightarrow \frac{x_{n+1} - x_n}{h} = -\frac{f(x)}{hg(x)} - \frac{hf'(x+h)}{hg(x)} - \epsilon(h) \xrightarrow{0} \text{Approaching the limit it comes:}$$

$$hx'f'(x) = -f(x) - hf'(x) \Rightarrow x' = -\frac{1}{h} \frac{f(x)}{g(x)} - h \Leftrightarrow \frac{dx}{df(x)} = -\frac{1}{h} \frac{f(x)}{g(x)} - 1 \Leftrightarrow$$

$$dx = \left(-\frac{1}{h} \frac{f(x)}{g(x)} - 1\right) dg(x) \Leftrightarrow$$

$$dx = -\left(\frac{1+h}{h}\right) \left(\frac{f(x)}{g(x)} + 1\right) g'(x) dx \Leftrightarrow x = -\int \frac{1}{h} \left\{f(x) \frac{g'(x)}{g(x)} - g'(x)\right\} dx \Leftrightarrow x = -\int \left(\frac{1+h}{h}\right) f(x) \ln g(x) dx - \int g'(x) dx \Leftrightarrow$$

$$x = -\int \left(\frac{1+h}{h}\right) f(x) \ln' f(x) dx - g(x) \quad (17)$$

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One can use our integral of the two-level formula in solving the problem and the program used in the routine presented, given that $x = \left(\frac{h+1}{h}\right) \int \ln g(x) dg(x)$. One could use the previous expression on the computer to compute the value of x by approximations.

CONCLUSION

The application of the Emanuel Vaz fractal geometry (E-K) to dipole, patch, and log-periodic antennas demonstrated significant potential for antenna miniaturization while maintaining acceptable electrical performance. The experimental and simulated results showed that the E-K geometry provides a reduction of approximately 32% in the overall element length and spacing when compared with equivalent conventional structures operating in the same frequency range.

This size reduction was verified in the E-3 centre-fed fractal dipoles and in the log-periodic fractal antennas designed for operation in the 470-834 MHz band. The results confirm that the increased electrical path introduced by the E-K geometry allows physically smaller radiating elements to achieve resonance at frequencies comparable to those of larger conventional antennas.

Good agreement was obtained between numerical simulations and experimental measurements, supporting the validity of the modelling approach used for the proposed fractal structures. The observed differences remained within the expected limits associated with fabrication tolerances, conductor characteristics, and measurement conditions.

The results further indicate that the design methodologies commonly applied to Koch-type fractal antennas can be adapted to the Emanuel Vaz geometry with appropriate modifications. Consequently, existing computational tools and optimisation procedures developed for related fractal antennas may be employed for the analysis and design of E-K fractal antennas.

Overall, the proposed Emanuel Vaz fractal geometry represents an effective alternative for antenna miniaturisation in applications where physical dimensions are constrained, particularly in the UHF band, while preserving radiation characteristics consistent with practical communication requirements.

Module NewtonRaphson

Function f(x As Double) As Double

' Define a função aqui

Return Math.Log(x) - 1

End Function

Function f_prime(x As Double) As Double

' Derivada da função

Return 1 / x

End Function

Sub Main()

Dim x0 As Double = 2.0 ' Chute inicial

Dim tolerance As Double = 0.0001

Dim maxIterations As Integer = 100

Dim x1 As Double

For i As Integer = 1 To maxIterations

x1 = x0 - f(x0) / f_prime(x0)

If Math.Abs(x1 - x0) < tolerance Then

Exit For

End If

x0 = x1

Next

Console.WriteLine("A raiz aproximada é: " & x1)

End Sub

End Module

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